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POSITIVE HARMONIC FUNCTIONS ON LIPSCHITZ DOMAINS

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0. **Introduction.** The results of this paper are based on a study of certain kernel functions associated with Lipschitz domains D. These functions are related to harmonic measure in D and to the ideal boundary of D as defined by R. S. Martin (see [6]), and are analogous to the Poisson kernel.

Let $D \subset E_{n+1}$ be a Lipschitz domain with a point P_0 fixed. We say that u is a kernel function at $Q_0 \in \partial D$ if $u(P)$ is positive and harmonic for $P \in D$ with $u(P_0) = 1$ and $u(P)$ vanishes continuously as $P \to Q$ for each $Q \in \partial D$, $Q \neq Q_0$.

One fundamental result of the paper is a uniform estimate for various approximations to kernel functions and another is the uniqueness of kernel functions. We use the uniform estimate to show that functions arising from several different constructions are in fact kernel functions, and the uniqueness then leads to further results.

The first application is to kernels related to harmonic measure. Recall that the generalized solution of the Dirichlet problem in D for continuous boundary values $f(Q)$ is

$$\int_{\partial D} f(Q)K(P, Q)\, d\omega^{P_0}(Q), \qquad P \in D,$$

where $\omega^P(\cdot)$ denotes harmonic measure in D and

$$K(P, Q) = \frac{d\omega^P}{d\omega^{P_0}}(Q).$$

For a Lipschitz domain D, we will show that $K(P, Q_0)$ is the unique kernel function at Q_0 and that $K(P, Q)$ is a continuous function of $Q \in \partial D$ for fixed $P \in D$. We will use these facts to discuss the general theory of R. S. Martin for Lipschitz domains.

The development originated by Martin is based on kernels which are limits of quotients of Green's functions. Martin uses these kernels to define an ideal boundary Δ of D and a corresponding topology on $D \cup \Delta$. For Lipschitz domains we will show that Martin's kernels are exactly the functions $K(P, Q)$. We can then identify the ideal boundary with the Euclidean boundary and show that the Martin topology

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is equivalent to the Euclidean topology. As a special case of a general theorem due to Martin, we obtain for each positive harmonic function u in D a unique Borel measure $d\mu$ on ∂D such that

$$u(P) = \int_{\partial D} K(P, Q)\, d\mu(Q), \qquad P \in D.$$

This representation and a classical Lebesgue point proof based on estimates of $K(P, Q)$ lead to the existence of finite nontangential limits for u at almost every $(d\omega^{P_0})$ point of ∂D. (See [5].) Viewing the representation in the general sense of Martin leads to the existence of finite fine limits for u almost everywhere on the ideal boundary. (See [4] or [8].) Having identified the ideal boundary with ∂D for Lipschitz domains, we can compare these notions of limit at the boundary. We will show that the existence of fine limits is equivalent to the existence of nontangential limits almost everywhere. We will also compare the pointwise existence of nontangential and semifine limits.

Our point of view is to use the classical results to illustrate the general theory. In particular, we introduce facts from the general theory only after they can be discussed in classical terms. The reader who is familiar with the general theory will easily recognize when we are proving a special case of a general result.

The style of this paper is similar to that of [5]. Proofs are again based on Harnack's principle, the maximum principle and the generalized solution of the Dirichlet problem. The crucial uniform estimate proved in §2 below is a generalization of a result from L. Carleson [3]. The motivation for our work came from Brelot and Doob [2] where relations between the classical and general theories are given for the special case $D=E^+_{n+1}$. The continuity of $K(\cdot, Q)$ for domains with boundary given by a function of class $C^{1+\varepsilon}$, $\varepsilon>0$, is proved for example in Widman [9]. Choquet has proved the continuity of $K(\cdot, Q)$ for domains with C^1 boundary.

1. **Definitions and preliminary facts.** We call an open bounded set $D\subset E_{n+1}$ a Lipschitz domain if D and ∂D are given locally by a function of class Lip 1—that is, to each point $Q_0 \in \partial D$ there corresponds a local coordinate system (X, y), $X \in E_n$, y real, and a function $b(X)$ such that

(i) $|b(X)-b(X')| \leqq C_0|X-X'|$ for all $X, X' \in E^n$ and

(ii) $D \cap N=\{(X, y) : y>b(X)\} \cap N$ for some neighborhood N of Q_0.

Clearly the same coordinate system (X, y) and function $b(X)$ can be used for all $Q \in \partial D \cap N$. We call the constant C_0 in (i) the local Lipschitz constant of D near Q_0. Although constants occurring in many inequalities depend on the local Lipschitz constant, our final results depend only on its existence, and we make no attempt to choose the local coordinate systems and neighborhoods so as to obtain the best constants.

Lipschitz domains are characterized by the existence of standard inner and outer cones. For fixed D, $Q_0 \in \partial D$, (X, y), $b(X)$, C_0 and N, consider the cones

$$\Gamma_+(X_1, y_1) = \{(X, y) : y-y_1 > C_0|X-X_1|\}$$

and

$$\Gamma_-(X_1, y_1) = \{(X, y) : y - y_1 < -C_0|X - X_1|\}.$$

If $(X_1, y_1) \in \bar{D} \cap N$ then $\Gamma_+(X_1, y_1) \cap N \subset D$, an observation we will use in applications of Harnack's inequality; while if $(X_1, y_1) \in N - D$ then

$$\Gamma_-(X_1, y_1) \cap N \subset N - \bar{D}.$$

The concept of nontangential limit is directly related to Harnack's principle. If $Q_0 = (X_0, y_0) \in \partial D$, we call a subset S of D a nontangential subset at Q_0 if there are constants $C, C' > 0$ such that $P \in S \cap \{P' : 2^{-k-1} \leqq |P' - Q_0| \leqq 2^{-k}\}$ implies

$$C' \leqq u(P)/u((X_0, y_0 + 2^{-k})) \leqq C$$

for all positive harmonic functions u in D. Using Harnack's principle and interior cones Γ_+, we see in particular that S is a nontangential subset of D at Q_0 if there is a constant $C > 0$ such that $\operatorname{dist}(P, \partial D) \geqq C|P - Q_0|$ for $P \in S$. We say a function $u(P)$ in D has nontangential limit L at $Q_0 \in \partial D$ if for every nontangential subdomain S at Q_0

$$\lim_{P \to Q_0; P \in S} u(P) = L.$$

We denote by $\omega^P(E)$ the harmonic measure of a Borel set $E \subset \partial D$—that is,

$$\omega^P(E) = \inf\left\{\psi(P) : \psi \text{ superharmonic in } D, \liminf_{P \to Q} \psi(P) \geqq \chi_E(Q)\right\},$$

where $\chi_E(Q)$, $Q \in \partial D$, is the characteristic function of E. For any such E, $\omega^P(E)$ is harmonic in D, and for any fixed P, $\omega^P(\cdot)$ is a Borel measure on ∂D. Given P and P_0, Harnack's principle implies that the measures $\omega^P(\cdot)$ and $\omega^{P_0}(\cdot)$ are absolutely continuous with respect to each other. Fixing P_0, we may then write

$$\frac{d\omega^P}{d\omega^{P_0}}(Q) = K(P, Q),$$

$P \in D$, in the Radon-Nikodym sense. Given $P \in D$, $K(P, Q)$ is therefore defined for almost every $(d\omega^{P_0})Q \in \partial D$.

Harmonic measure is related to the generalized solution of the Dirichlet problem. If $f(Q)$, $Q \in \partial D$, is integrable with respect to $d\omega^{P_0}$, the generalized solution of the Dirichlet problem with boundary values f is

$$\inf\left\{\psi(P) : \psi \text{ superharmonic in } D, \liminf_{P \to Q} \psi(P) \geqq f(Q)\right\}$$

$$= \int_{\partial D} f(Q)\, d\omega^P(Q) = \int_{\partial D} f(Q) K(P, Q)\, d\omega^{P_0}(Q).$$

Lipschitz domains are regular for the solution of the Dirichlet problem since each point of ∂D satisfies the Poincaré cone condition—that is, for $Q \in \partial D$ the cone $\Gamma_-(Q) \cap N$ is contained in the complement of D. It follows that

$$\lim_{P \to Q_0} \int_{\partial D} f(Q) K(P, Q)\, d\omega^{P_0}(Q) = f(Q_0)$$

at each point Q_0 of continuity of f. Hence if $u(P)$ is harmonic in D and continuous in $\bar{D}$,

$$u(P) = \int_{\partial D} u(Q)K(P, Q)\, d\omega^{P_0}(Q), \qquad P \in D.$$

For $Q_0 = (X_0, y_0) \in \partial D$, we will denote by $\psi(Q_0, r, s)$ the cylinder

$$\{(X, y) : |X - X_0| < r, |y - y_0| < s\}.$$

We will use only cylinders of the form $\psi(Q_0, r, rs)$ where s is sufficiently large to insure that the top of the cylinder is contained in D and the bottom is contained in the complement of D for all small $r > 0$. The choice of s depends only on the local Lipschitz constant. We denote $\Delta(Q_0, r) = \partial D \cap \psi(Q_0, r, s)$ and call Δ a disc.

Locally we may project ∂D onto the n-dimensional hyperplane $\{(X, 0)\}$. Hence any Borel measure on ∂D may be considered as one on E_n. This allows us to apply Besicovitch's theorem on the differentiation of measures on E_n. (See [1].) We obtain the following theorem.

THEOREM. *Suppose D is a Lipschitz domain and μ and ν are finite Borel measures on ∂D. If $d\mu = f\, d\nu + ds$ where s is singular with respect to ν, then for almost every* $(d\nu)$, $Q_0 \in \partial D$,

$$\frac{1}{\nu(\Delta(Q_0, r))} \int_{\Delta(Q_0, r)} |f(Q) - f(Q_0)|\, d\nu(Q) \to 0$$

and

$$\mu(\Delta(Q_0, r))/\nu(\Delta(Q_0, r)) \to f(Q_0)$$

as $r \to 0$.

Applying this result to the definition of $K(P, Q)$, we see that for each $P \in D$

$$K(P, Q_0) = \lim_{r \to 0} \frac{\omega^P(\Delta(Q_0, r))}{\omega^{P_0}(\Delta(Q_0, r))}$$

for almost every $(d\omega^{P_0})Q_0 \in \partial D$. It follows in particular that $K(P, Q)$ can be estimated in terms of $\omega^P(\Delta)$.

2. **A uniform estimate.** In this section we obtain a uniform estimate for various approximations to kernel functions on Lipschitz domains. At the end of the section we indicate how this estimate, together with the uniqueness of kernel functions, can be used to obtain further results.

The uniform estimate is given in (2.4) below and the crucial step in deriving it is Lemma (2.2). Both Lemmas (2.1) and (2.2) below are essentially due to L. Carleson. (See [3], or [5].) Here, however, they are in a form somewhat different from that in [5], and we shall repeat their proofs.

LEMMA (2.1). *If D is a Lipschitz domain there is a constant $C > 0$ such that*

$$\omega^P(\Delta(Q_0, 2r)) \geqq C, \qquad P \in \psi(Q_0, r, rs).$$

C depends only on s and the Lipschitz constant of D near Q_0.

Proof. Let $h(P)$, $P \in \psi(Q_0, 2r, 2rs)$, be the harmonic measure of the bottom of $\psi(Q_0, 2r, 2rs)$. Since the bottom of the cylinder is outside D, the maximum principle implies

$$\omega^P(\Delta(Q_0, 2r)) \geqq h(P), \qquad P \in D \cap \psi(Q_0, 2r, 2rs).$$

The lemma follows if we choose $C = \inf\{h(P) : P \in \psi(Q_0, r, rs)\}$. For then $C > 0$ and C depends only on the proportions of the cylinder.

LEMMA (2.2). *Suppose D is a Lipschitz domain and A is the point in the center of the top of the cylinder $\psi(Q_0, r, rs)$. (If $Q_0 = (X_0, y_0)$ then $A = (X_0, y_0 + rs)$.) Then there is a constant $C > 0$ such that for any disc $\Delta' = \Delta(Q_0, r')$, $0 < r' < r$,*

$$\omega^P(\Delta') \leqq C\omega^A(\Delta')$$

for $P \in D - \psi(Q_0, r, rs)$. C depends only on s and the Lipschitz constant of D near Q_0.

Proof. Since $\omega^P(\Delta')$ vanishes on $\partial D - \psi(r, rs)$, it is enough by the maximum principle to prove the result for $P \in D \cap \partial\psi(r, rs)$. We note that if $P \in D \cap \partial\psi(r, rs)$ is away from ∂D then $\omega^P(\Delta') \leqq C\omega^A(\Delta')$ by Harnack's principle. If $P \in D \cap \partial\psi(r, rs)$ is near ∂D then $\omega^P(\Delta')$ is small, and therefore $\omega^P(\Delta') \leqq C\omega^A(\Delta')$. We must find a constant C independent of r and r'.

To accomplish this we introduce an auxiliary function $h(P)$. Let $X_2 = (2, 0, \ldots, 0) \in E_n$ and $P_2 = (X_2, 0) \in E_{n+1}$. Let $\tilde{D}$ be the intersection of the exterior of the cylinder $\{(X, y) : |X| \leqq 1, |y| \leqq s_0\}$ and the exterior of the truncated cone

$$\{(X, y) : -s_0 \leqq y \leqq -M_0|X - X_2|\}.$$

Let $h_N(P)$ be the harmonic measure in $\tilde{D} \cap \{(X, y) : |(X, y)| < N\}$ of the part of the boundary on the cylinder. Then $h_N(P)$ increases to a harmonic function $h(P)$ in $\tilde{D}$. Clearly $h(P)$ has boundary values 1 on the cylinder. Since the harmonic measure of the cylinder and the outer sphere majorizes $h(P)$ in $\tilde{D} \cap \{(X, y) : |(X, y)| < N\}$, we see that $h(P) \to 0$ as $P \to P_2$.

If $\Delta' = \Delta(Q_0, r')$, let A_n denote the point in the middle of the top of

$$\psi_n = \psi(Q_0, 2^{n-1}r', 2^{n-1}r's), \qquad n = 1, 2, \ldots, N,$$

where $2^{N-1}r' < r \leqq 2^N r'$. By Harnack's principle, there is a constant $C_1 > 0$ such that $\omega^{A_n}(\Delta') \leqq C_1\omega^{A_{n+1}}(\Delta')$.

Choose y_0 such that $h(X_2, y) < 1/C_1$ for $0 < y < y_0$. Let β_n be the points on $\gamma_n = D \cap \partial\psi_n$ which are less than $2^{n-1}r'y_0$ above ∂D and let $\alpha_n = \gamma_n - \beta_n$. By Harnack's principle there is a constant $C_2 > 0$ such that $\omega^P(\Delta') \leqq C_2\omega^{A_n}(\Delta')$ for $P \in \alpha_n$.

By Lemma (2.1) there is a constant $C_3 > 0$ such that $\omega^{A_1}(\Delta') \geqq C_3$. Hence

$$\omega^P(\Delta') \leqq 1 \leqq (1/C_3)\omega^{A_1}(\Delta'), \qquad P \in D.$$

If $C_4 = \max[C_2, 1/C_3]$, then $\omega^P(\Delta') \leqq C_4\omega^{A_1}(\Delta')$ for $P \in D$ and in particular for $P \in \gamma_1$. Also, $\omega^P(\Delta') \leqq C_4\omega^{A_2}(\Delta')$ for $P \in \alpha_2$, and we will show the same is true for $P \in \beta_2$.

For any $P \in \beta_2$, shrink $\tilde{D}$ by the factor r' and position it so that the point P_2 is on ∂D below P and the axis of the cylindrical part of $r'\tilde{D}$ coincides with the axis of ψ_1. If we choose M_0 larger than the local Lipschitz constant M of D near Q_0, then the conical part of $r'\tilde{D}$ near Q_0 will be outside D. If we choose $s_0 \geqq s+2M$, the cylindrical part of $r'\tilde{D}$ will contain ψ_1.

Since $\omega^P(\Delta') \leqq C_4\omega^{A_1}(\Delta')$ for $P \in \gamma_1$, the maximum principle implies that

$$\omega^P(\Delta') \leqq C_4\omega^{A_1}(\Delta')h'(P), \qquad P \in D-\psi_1,$$

where h' in $r'\tilde{D}$ corresponds to h in $\tilde{D}$.

Therefore

$$\omega^P(\Delta') \leqq C_4C_1\omega^{A_2}(\Delta')h'(P), \qquad P \in D-\psi_1,$$

and since $h'(P) < 1/C_1$ for $P \in \beta_2$,

$$\omega^P(\Delta') \leqq C_4\omega^{A_2}(\Delta'), \qquad P \in \gamma_2.$$

The step from $n=1$ to $n=2$ is typical. Continuing the process, we see that

$$\omega^P(\Delta') \leqq C_4\omega^{A_N}(\Delta') \qquad \text{for } P \in \gamma_N,$$

and therefore for $P \in D-\psi(r, rs)$. By Harnack's principle there is a constant C such that $\omega^{A_N}(\Delta') \leqq c\omega^A(\Delta')$, and the lemma follows.

We will use Lemma (2.2) to obtain an estimate on $K(P, Q)$ as a function of P uniformly in Q. Let A be the point in the center of the top of $\psi(Q_0, r, rs)$. If $Q \in \Delta(Q_0, r)$ and Δ' is a small disc with center Q, we obtain from Lemma (2.2) and Harnack's principle that

$$\omega^P(\Delta') \leqq C\omega^A(\Delta'), \qquad P \in D-\psi(Q_0, 2r, 2rs).$$

Since

$$K(P, Q) = \lim_{\Delta' \to Q} \frac{\omega^P(\Delta')}{\omega^{P_0}(\Delta')}$$

for almost every Q, we see that for any fixed $P \in D-\psi(Q_0, 2r, 2rs)$, $K(P, Q) \leqq CK(A, Q)$ for almost every $Q \in \Delta(Q_0, r)$.

Now let u be harmonic and positive in D, continuous in $\bar{D}$ and zero on ∂D except possibly in $\Delta(Q_0, r)$. Then

$$u(P) = \int_{\Delta(Q_0,r)} K(P, Q)u(Q)\, d\omega^{P_0}(Q).$$

Hence by our remarks above,

$$u(P) \leqq Cu(A), \qquad P \in D-\psi(Q_0, 2r, 2rs). \tag{2.3}$$

From Lemma (2.1) and the maximum principle we obtain

$$u(P) \leqq Cu(A)\omega^P(\Delta(Q_0, 4r)), \qquad P \in D-\psi(Q_0, 2r, 2rs).$$

We will need this result for a larger class of functions u. We can accomplish this by making a simple observation. Suppose N is a neighborhood of Q_0 such that

$N \subset \psi(Q_0, r/4, rs/4)$ and $D-N$ is a Lipschitz domain with local Lipschitz constant less than a fixed multiple of that of D. (N may be, for example, a truncated cone which opens downward.) Applying (2.3) to $D-N$ we obtain

$$u(P) \leqq Cu(A)\omega^P(\Delta), \qquad P \in D-\psi, \tag{2.4}$$

where C depends only on s and the Lipschitz constant of D near Q_0, $\Delta = \Delta(Q_0, r)$, $\psi = \psi(Q_0, r, rs)$, A is the point above Q_0 with $|A - Q_0| = rs$, and u is any function which satisfies

$$\begin{aligned} &u \text{ is harmonic and positive in } D-N, \\ &u \text{ is continuous in } \overline{D-N} \text{ and} \\ &u(Q) = 0 \quad \text{for } Q \in \partial D - N. \end{aligned} \tag{2.5}$$

LEMMA (2.6). *Suppose that u_n satisfies* (2.5) *and that the corresponding neighborhoods N_n shrink to Q_0 as n approaches infinity. If $u_n(P_0) = 1$ for all n and $u_n(P)$ approaches a harmonic function $u(P)$ then u is a kernel function at Q_0.*

Proof. We need only show that u vanishes continuously at each $Q \in \partial D$, $Q \neq Q_0$. Note however that for any fixed choice of N, Δ, ψ and A, (2.4) holds uniformly for all u_n with n sufficiently large.

We now give various applications of Lemma (2.6) which prove the existence of kernel functions at each $Q_0 \in \partial D$ and illustrate the significance of having only one kernel function at each Q_0.

REMARK (2.7). Suppose

$$u_n(P) = \omega^P(\Delta(Q_0, 2^{-n}))/\omega^{P_0}(\Delta(Q_0, 2^{-n}))$$

converges to a harmonic function $K(P, Q_0)$ as $n \to \infty$. Then $K(P, Q_0)$ is a kernel function at Q_0. Note that a subsequence of u_n does converge to a harmonic function, so there is at least one kernel function at Q_0. If there is only one kernel function at Q_0 then the functions u_n must converge as $n \to \infty$.

REMARK (2.8). Suppose u_n is a kernel function at Q_n and $Q_n \to Q_0$ as $n \to \infty$. If u_n converges to a harmonic function u then u is a kernel function at Q_0. A subsequence of u_n does converge to a harmonic function. Hence if there is only one kernel function $K(P, Q)$ at each Q, $K(P, Q_n)$ converges to $K(P, Q_0)$—i.e., $K(P, Q)$ is a continuous function of Q.

REMARK (2.9). Let

$$u_n(P) = G(P, M_n)/G(P_0, M_n)$$

where $G(P, M)$ denotes the Green's function of D with pole at M. If $M_n \to Q_0$ as $n \to \infty$ and u_n converges to a harmonic function u, then u is a kernel function at Q_0. It follows that the uniqueness of kernel functions will allow us to identify Martin's kernels with the functions $K(P, Q)$.

3. **Uniqueness of kernel functions.** We will need an inequality opposite to (2.4). We will derive this inequality when D is a starlike Lipschitz domain—that is, D

is a Lipschitz domain which is starlike about P_0 and the y-axis of a local coordinate system at any $Q \in \partial D$ may be chosen to contain the segment QP_0. (For any point $Q \in \partial D$, $\psi(Q, r, s) \cap D$ will be a starlike Lipschitz domain with center near the top of the cylinder if r is sufficiently small.)

LEMMA (3.1). *Suppose D is a starlike Lipschitz domain with center $P_0 = 0$. For $Q_0 \in \partial D$ let $\psi = \psi(Q_0, r, rs)$ and $\Delta = \Delta(Q_0, r)$. Let A denote the point in the center of the top of ψ and let t, $0 < t < 1$, be such that $tQ_0 = A$. If u is positive and harmonic in D then*

$$u(tP) \geqq Cu(A)\omega^P(\Delta), \qquad P \in D,$$

where $C > 0$ depends only on the local Lipschitz constant.

Proof. For s sufficiently large Harnack's inequality implies there is a constant $C > 0$ such that $u(tQ) \geqq Cu(tQ_0) = Cu(A)$ for $Q \in \Delta$. Then

$$\begin{aligned} u(tP) &= \int_{\partial D} u(tQ)K(P, Q)\, d\omega^{P_0}(Q) \\ &\geqq \int_{\Delta} u(tQ)K(P, Q)\, d\omega^{P_0}(Q) \\ &\geqq Cu(A)\omega^P(\Delta). \end{aligned}$$

LEMMA (3.2). *If D is a starlike Lipschitz domain with center $P_0 = 0$ there are positive constants C and C' such that*

$$C' \leqq u(P)/v(P) \leqq C, \qquad P \in D,$$

for all kernel functions u, v at Q_0.

Proof. Let ψ, Δ, A and t be as in Lemma (3.1). Setting $P = P_0$ in (3.1) and (2.4) we obtain $1 = u(P_0) \geqq Cu(A)\omega^{P_0}(\Delta)$ and $1 = u(P_0) \leqq Cu(A)\omega^{P_0}(\Delta)$, respectively. Combining these inequalities with (3.1) and (2.4) we obtain

$$u(tP) \geqq C\omega^P(\Delta)/\omega^{P_0}(\Delta), \qquad P \in D,$$

and

$$u(P) \leqq C\omega^P(\Delta)/\omega^{P_0}(\Delta), \qquad P \in D - \psi.$$

Choose a sequence $r_n \to 0$ such that

$$\lim_{n \to \infty} \frac{\omega^P(\Delta(Q_0, r_n))}{\omega^{P_0}(\Delta(Q_0, r_n))} = h(P)$$

is harmonic in D. Noting that $r_n \to 0$ implies the corresponding numbers $t_n \to 1$ we obtain

$$C'h(P) \leqq u(P) \leqq Ch(P), \qquad P \in D.$$

This proves the lemma.

Let us note here that if we assumed Martin's representation theorem and various results from the general theory we could use Lemma (3.2) to prove the uniqueness of kernel functions. We choose instead to prove our results without appealing to the general theory.

Suppose that B is a relatively closed subset of D and u is nonnegative and superharmonic in D. Define

$$R_u^B(P) = \inf\{\psi(P) : \psi \text{ nonnegative and superharmonic in } D \text{ with } \psi \geqq u \text{ on } B\}.$$

This concept of "reduced function" is basic to the theory.

Clearly, $0 \leqq R_u^B \leqq u$ in D and $R_u^B = u$ on B, $R_{cu}^B = cR_u^B$ for $c \geqq 0$, $R_u^{A \cup B} \leqq R_u^A + R_u^B$, $R_{u+v}^B \leqq R_u^B + R_v^B$, $R_u^A \leqq R_u^B$ if $A \subset B$ and $R_u^B \leqq R_v^B$ if $u \leqq v$ on B. In general, R_u^B must be modified on a small subset of B in order to obtain a superharmonic function. (See [6] or [7].) However, we will only apply the definition when the nonnegative, superharmonic function u is continuous in D and $D-B$ is regular for the solution of the Dirichlet problem. In this case R_u^B in $D-B$ is the generalized solution of the Dirichlet problem with boundary values 0 on ∂D and $u(P)$ for $P \in \partial B \cap D$. It follows that R_u^B is continuous and superharmonic in D.

We will be interested in a particular type of set B. Given a sequence of points $P_k \in D$ which approach $Q_0 \in \partial D$ nontangentially, choose $r_0 > 0$ such that the closure of each ball $B(P_k, r_0|P_k - Q_0|)$ with center P_k and radius $r_0|P_k - Q_0|$ lies in D. For $0 < r < r_0$ let $B_k = B(P_k, r|P_k - Q_0|)$. We then call the set $B = \bigcup_{k=1}^{\infty} B_k$ a n.t. $\mathscr{B}$ set at Q_0. Note that by Harnack's inequality there is a constant $C(r/r_0) > 1$ such that

$$[C(r/r_0)]^{-1} \leqq u(P_k)/u(P) \leqq C(r/r_0), \qquad P \in B_k,$$

for all positive harmonic functions in D. Also, $C(r/r_0) \to 1$ as $r/r_0 \to 0$.

If B is any n.t. $\mathscr{B}$ set at Q_0 and u is a kernel function at Q_0 we will show that $R_u^B = u$. This means essentially that $u(P)$ is determined in D by its values on any sequence which converges nontangentially to Q_0.

The following two lemmas are useful.

Lemma (3.3). *If u is a kernel function at Q_0 then $R_u^B = u$ if and only if*

$$\limsup_{P \to Q_0} [u(P) - R_u^B(P)] = 0.$$

Proof. This follows immediately from the fact that $u - R_u^B$ is subharmonic and nonnegative in D and vanishes continuously at $Q \in \partial D$, $Q \neq Q_0$.

Lemma (3.4). *Suppose v is harmonic in D, u is superharmonic and continuous in D and $0 \leqq v \leqq u$. If $R_u^B = u$, then $R_v^B = v$.*

Proof. Consider $\psi = u - [v - R_v^B]$. ψ is nonnegative and superharmonic in D and $\psi = u$ on B. Hence, $R_u^B \leqq \psi$. But if $R_v^B < v$ then $R_u^B \leqq \psi < u$. This proves the lemma.

Suppose that for an arbitrary n.t. $\mathscr{B}$ set B at Q_0 there is one kernel function u with $R_u^B = u$. If D is a starlike Lipschitz domain we see from Lemmas (3.2) and (3.4) that $R_v^B = v$ for all kernel functions v at Q_0.

LEMMA (3.5). *Suppose D is a starlike Lipschitz domain with center $P_0=0$ and B is a n.t. $\mathscr{B}$ set at Q_0. Then there is a kernel function u at Q_0 with $R_u^B=u$. Hence, $R_u^B=u$ if B is any n.t. $\mathscr{B}$ set at Q_0 and u is any kernel function at Q_0.*

Proof. Let u_0 be any kernel function at Q_0. (We know u_0 exists by Remark (2.7).) If $B=\bigcup_{n=1}^{\infty} B_n$, let $u_k=R_{u_0}^{\bigcup_{n=k}^{\infty} B_n}$. Then u_k decreases to a harmonic function h in D. Since $h\leqq u_0$, h vanishes continuously at each $Q\in\partial D$, $Q\neq Q_0$. Since $u_1=u_0$ on B we have $R_{u_1}^B=R_{u_0}^B=u_1$. Also, $0\leqq h\leqq u$, and therefore $R_h^B=h$ by Lemma (3.4). Hence, if $h(P_0)\neq 0$, $u(P)=h(P)/h(P_0)$ will be the desired kernel function at Q_0.

To show that $h(P_0)>0$ we note that $u_k(P)\geqq R_{u_0}^{B_k}(P)$, where $B_k=B(P_k, r|P_k-Q_0|)$. Choose t_k, $0<t_k<1$, such that $t_kQ_k=P_k$ for some $Q_k\in\partial D$ and let

$$\delta_k = \{Q\in\partial D : t_kQ\in B_k\}.$$

Since $R_{u_0}^{B_k}(P)=u_0(P)$ for $P\in B_k$, Harnack's principle implies $R_{u_0}^{B_k}(P)\geqq Cu_0(P_k)$, $P\in B_k$. The minimum principle for superharmonic functions then implies $R_{u_0}^{B_k}(t_kP)\geqq Cu_0(P_k)\omega^P(\delta_k)$, $P\in D$. In particular, $R_{u_0}^{B_k}(P_0)\geqq Cu_0(P_k)\omega^{P_0}(\delta_k)$. We will show there exists $C>0$ such that $u_0(P_k)\omega^{P_0}(\delta_k)\geqq C$ for all k. To see this let Δ_k denote the smallest disc with center Q_0 which contains δ_k in its middle fourth. Since the radius of Δ_k is proportional to $|P_k-Q_0|$ (uniformly in k) Harnack's principle allows us to substitute P_k for A in (2.4) and Lemma (3.1) to obtain $1=u_0(P_0)\leqq Cu_0(P_k)\omega^{P_0}(\Delta_k)$ and $\omega^{P_0}(\delta_k)\geqq C\omega^{P_k}(\delta_k)\omega^{P_0}(\Delta_k)$, respectively. But the Lipschitz character of ∂D implies that δ_k contains a disc $\Delta(Q_k, r_1|P_k-Q_k|)$, so Lemma (2.1) implies $\omega^{P_k}(\delta_k)\geqq C$. Combining results we obtain the lemma.

We can easily remove the assumption that D is starlike.

LEMMA (3.6). *Suppose D is a Lipschitz domain, B is any n.t. $\mathscr{B}$ set at Q_0 and u is any kernel function at Q_0. Then $R_u^B=u$.*

Proof. Suppose $R_u^B<u$ at some point of D. Consider the starlike Lipschitz domain $\tilde{D}=\psi(Q_0, r, s)\cap D$, r small. Let $\tilde{R}$ and $\tilde{\omega}^P(\cdot)$ in $\tilde{D}$ correspond to R and $\omega^P(\cdot)$ in D and let $\gamma=D\cap\partial\tilde{D}$. Since u is continuous on $\partial\tilde{D}$ except at Q_0, a suitable normalization of

$$v(P) = u(P)-\int_\gamma u(Q)\,d\tilde{\omega}^P(Q), \qquad P\in\tilde{D},$$

is a kernel function at Q_0 for $\tilde{D}$. Hence, $\tilde{R}_v^B=v$. (We may assume $B\subset\tilde{D}$.)

We assert that $R_u^B<u$ implies $R_u^{B\cup(D-\tilde{D})}<u$. For since $R_u^{B\cup(D-\tilde{D})}\leqq R_u^B+R_u^{(D-\tilde{D})}$, $u-R_u^{B\cup(D-\tilde{D})}\geqq u-R_u^B-R_u^{(D-\tilde{D})}$, and therefore

$$\begin{aligned}\limsup_{P\to Q_0}\,[u-R_u^{B\cup(D-\tilde{D})}] &\geqq \limsup_{P\to Q_0}\,[u-R_u^B-R_u^{(D-\tilde{D})}]\\ &= \limsup_{P\to Q_0}\,[u-R_u^B].\end{aligned}$$

Our assertion now follows from Lemma (3.3).

We now show that $R_u^{B\cup(D-\tilde{D})} < u$ implies $\tilde{R}_v^B < v$. To see this, consider

$$\psi(P) = R_u^{B\cup(D-\tilde{D})}(P) - \int_\gamma u(Q)\, d\tilde{\omega}^P(Q), \qquad P \in \tilde{D}.$$

ψ is nonnegative and superharmonic in $\tilde{D}$ and $\psi = v$ on B. Hence $\tilde{R}_v^B \leqq \psi < v$, and this contradiction proves the lemma.

We can now prove the uniqueness of kernel functions. The proof illustrates our earlier remark that kernel functions are determined in D by their values on any sequence of points in D which approach Q_0 nontangentially.

LEMMA (3.7). *If D is a Lipschitz domain there is exactly one kernel function at each $Q_0 \in \partial D$.*

Proof. Let u and v be any two kernel functions at Q_0 and suppose $P_k \to Q_0$ nontangentially. We assert that

$$\lim_{k\to\infty} \frac{u(P_k)}{v(P_k)} = 1.$$

For suppose

$$\limsup_{k\to\infty} \frac{u(P_k)}{v(P_k)} > 1.$$

We can then choose $\varepsilon > 0$ and a subsequence $\{P_{k_j}\}_{j\geqq 1}$ such that $u(P_{k_j}) > (1+2\varepsilon)v(P_{k_j})$ for all j. Using Harnack's inequality we can then choose a n.t. $\mathscr{B}$ set B at Q_0, $B = \bigcup_{j=1}^\infty B(P_{k_j}, r|P_{k_j} - Q_0|)$, $r = r(\varepsilon)$, such that $u(P) > (1+\varepsilon)v(P)$ for $P \in B$. Then

$$u(P) = R_u^B(P) \geqq (1+\varepsilon)R_v^B(P) = (1+\varepsilon)v(P), \qquad P \in D.$$

Setting $P = P_0$ we obtain

$$1 = u(P_0) \geqq (1+\varepsilon)v(P_0) = 1+\varepsilon,$$

a contradiction. Similarly,

$$\liminf_{k\to\infty} \frac{u(P_k)}{v(P_k)} \geqq 1$$

and the assertion follows.

Now let $\{P_k\}$ be any fixed sequence which approaches Q_0 nontangentially. Given $\varepsilon > 0$ there is an N such that

$$(1-2\varepsilon) \leqq u(P_k)/v(P_k) \leqq (1+2\varepsilon)$$

for all $k \geqq N$. Harnack's inequality can then be used to construct a n.t. $\mathscr{B}$ set $B = \bigcup_{k=N}^\infty B(P_k, r|P_k - Q_0|)$ such that

$$(1-\varepsilon)v(P) \leqq u(P) \leqq (1+\varepsilon)v(P), \qquad P \in B.$$

Hence, for all $P \in D$,

$$\begin{aligned}(1-\varepsilon)v &= (1-\varepsilon)R_v^B \leqq R_u^B = u \\ &= R_u^B \leqq (1+\varepsilon)R_v^B = (1+\varepsilon)v.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $u(P) = v(P)$ for all $P \in D$.

4. **Kernel functions and a representation theorem.** The first application of our fundamental result concerns the Radon-Nikodym derivative

$$K(P, Q) = \frac{d\omega^P}{d\omega^{P_0}}(Q).$$

Recall that

$$K(P, Q) = \lim_{r \to 0} \frac{\omega^P(\Delta(Q, r))}{\omega^{P_0}(\Delta(Q, r))}$$

for a.e. $(d\omega^{P_0})Q \in \partial D$.

THEOREM (4.1). *If D is a Lipschitz domain there exists*

$$\lim_{r \to 0} \frac{\omega^P(\Delta(Q, r))}{\omega^{P_0}(\Delta(Q, r))} = K(P, Q)$$

for every $Q \in \partial D$. For fixed $Q \in \partial D$, $K(P, Q)$ is a positive and harmonic function of P in D and for fixed $P \in D$, $K(P, Q)$ is a continuous function of Q on ∂D.

Proof. In view of the uniqueness of kernel functions this theorem is a consequence of Remarks (2.7) and (2.8).

We can now identify the Martin boundary and topology with their classical analogues. We begin by recalling the definitions of the Martin boundary and related concepts.

For $M, P \in D$ let

$$\mathscr{K}(P, M) = G(P, M)/G(P_0, M),$$

where $G(P, M)$ is the Green's function for D with pole at M. Consider the family of sequences $\{M_k\}_{k \geq 1}$ of points of D without accumulation points in D for which $\mathscr{K}(P, M_k)$ converges in D to a harmonic function, denoted $\mathscr{K}(P, \{M_k\}_{k \geq 1})$. Two such sequences $\{M_k\}$ and $\{M_k'\}$ are called equivalent if $\mathscr{K}(P, \{M_k\}) = \mathscr{K}(P, \{M_k'\})$ and each equivalence class is called an element of the Martin boundary Δ. If M is such an equivalence class (i.e., $M \in \Delta$) then $\mathscr{K}(P, M)$ will denote the corresponding harmonic limit function. Thus each $M \in D \cup \Delta$ is associated with a unique function $\mathscr{K}(P, M)$. The Martin topology on $D \cup \Delta$ is given by the metric

$$\rho(M, M') = \int_\Sigma \frac{|\mathscr{K}(P, M) - \mathscr{K}(P, M')|}{1 + |\mathscr{K}(P, M) - \mathscr{K}(P, M')|} \, dP, \qquad M, M' \in D \cup \Delta,$$

where Σ is a small ball with center P_0. In general, $\mathscr{K}(P, M)$ is a ρ-continuous function of $M \in \Delta$ for $P \in D$ fixed, $D \cup \Delta$ is compact and complete with respect to ρ, $D \cup \Delta$ is the ρ-closure of D and the ρ-topology is equivalent to the Euclidean topology in D. (See [6].) We have the following results.

THEOREM (4.2). *If D is a Lipschitz domain there is a one-to-one correspondence between the Martin boundary of D and the Euclidean boundary. If $M \in \Delta$ corresponds to $Q \in \partial D$ then $\mathscr{K}(P, M) = K(P, Q)$. The Martin topology on $D \cup \Delta$ is equivalent to the Euclidean topology on $D \cup \partial D$.*

Proof. In view of the uniqueness of kernel functions Remark (2.9) shows that

$$\lim_{k\to\infty} \mathscr{K}(P, M_k) = K(P, Q)$$

if $\{M_k\}$ is any sequence of points in D such that $M_k \to Q \in \partial D$. It follows that each point of ∂D may be associated with a point of Δ. (2.4) clearly shows that $K(P, Q) \neq K(P, Q')$ if $Q \neq Q'$. Hence, the functions $\mathscr{K}(P, M_k)$ cannot converge if the sequence $\{M_k\}$ has more than one accumulation point on ∂D and different points of ∂D must be associated with different points of Δ. This gives a one-to-one correspondence between ∂D and Δ with $\mathscr{K}(P, M) = K(P, Q)$ when $M \in \Delta$ corresponds to $Q \in \partial D$.

Denote $\mathscr{K}(P, M)$ by $K(P, M)$ for $M \in D \cup \Delta$. If $M_k \to M$ in the Euclidean topology then $K(P, M_k) \to K(P, M)$ and, therefore, $M_k \to M$ in the ρ-topology by Lebesgue's dominated convergence theorem. On the other hand suppose $M_k \to M$ in the ρ-topology. If M_k does not converge to M in the Euclidean topology there is a subsequence M_{k_j} such that $M_{k_j} \to M' \neq M$ in the Euclidean topology. Then $M_{k_j} \to M'$ and $M_{k_j} \to M$ in the ρ-topology with $M' \neq M$, which is impossible. Therefore, the Martin ρ-topology on $D \cup \Delta$ is equivalent to the Euclidean topology on $D \cup \partial D$.

We conclude the section with a special case of Martin's representation theorem and a discussion of the general case.

THEOREM (4.3). *Suppose u is positive and harmonic in the Lipschitz domain D. Then there is a unique Borel measure $\mu(\cdot)$ on ∂D such that*

$$u(P) = \int_{\partial D} K(P, Q)\, d\mu(Q).$$

Proof. For a closed subset F of ∂D let

$$\mu_u^P(F) = \mu^P(F) = \inf\{R_u^{\bar{G}\cap D}(P) : G \text{ open in } E_{n+1},\ G \supset F\}.$$

(In the general case considered by Martin F is a subset of Δ and the terms open and closed refer to the ρ-topology.) The set function $\mu^P(F)$ defines a regular Borel measure on ∂D for each fixed $P \in D$. Since $\mu^P(F)$ is a positive harmonic function in D the measures μ^P are absolutely continuous with respect to μ^{P_0} by Harnack's inequality. Hence,

$$\mu^P(F) = \int_F d\mu^P(Q) = \int_F \frac{d\mu^P}{d\mu^{P_0}}(Q)\, d\mu^{P_0}(Q).$$

We assert that $d\mu^P(Q)/d\mu^{P_0} = K(P, Q)$ for a.e. $(d\mu^{P_0})$ $Q \in \partial D$. For by Besicovitch's theorem,

$$\frac{d\mu^P}{d\mu^{P_0}}(Q) = \lim_{r\to 0} \frac{\mu^P(\Delta(Q, r))}{\mu^{P_0}(\Delta(Q, r))}$$

for a.e. $(d\mu^{P_0})$ $Q \in \partial D$. Since the functions

$$\frac{\mu^P(\Delta(Q, r))}{\mu^{P_0}(\Delta(Q, r))}$$

satisfy the conditions of Lemma (2.6), their limit exists and is $K(P, Q)$. Hence

$$\mu^P(A) = \int_A K(P, Q)\, d\mu^{P_0}(Q)$$

for all Borel sets A and in particular,

$$u(P) = \mu^P(\partial D) = \int_{\partial D} K(P, Q)\, d\mu^{P_0}(Q).$$

Suppose now that

$$u(P) = \int_{\partial D} K(P, Q)\, d\nu(Q)$$

for a regular Borel measure ν on ∂D. For closed $F \subset \partial D$ we will show that $\nu(F) = \mu^{P_0}(F)$. Choose a sequence $\{G_k\}$ of open sets in E_{n+1} such that $F = \bigcap_{k=1}^{\infty} G_k$ and

$$\mu^{P_0}(F) = \lim_{k\to\infty} R_u^{\bar{G}_k \cap D}(P_0).$$

Let $\omega_k^P(\cdot)$ denote harmonic measure on $D - \bar{G}_k$. (We may assume $D - \bar{G}_k$ is regular.) Then for $P \in D - \bar{G}_k$,

$$\begin{aligned} R_u^{\bar{G}_k \cap D}(P) &= \int_{D\cap\partial\bar{G}_k} u(Q)\, d\omega_k^P(Q) \\ &= \int_{D\cap\partial\bar{G}_k} \left[\int_{\partial D} K(Q, Q')\, d\nu(Q')\right] d\omega_k^P(Q) \\ &= \int_{\partial D} \left[\int_{D\cap\partial\bar{G}_k} K(Q, Q')\, d\omega_k^P(Q)\right] d\nu(Q'). \end{aligned}$$

If $Q' \in F$ then

$$\int_{D\cap\partial\bar{G}_k} K(Q, Q')\, d\omega_k^P(Q) = K(P, Q').$$

If $Q' \notin F$ then

$$\lim_{k\to\infty} \int_{D\cap\partial\bar{G}_k} K(Q, Q')\, d\omega_k^P(Q) = 0.$$

Since

$$\int_{D\cap\partial\bar{G}_k} K(Q, Q')\, d\omega_k^P(Q) \leqq K(P, Q')$$

we may apply Lebesgue's dominated convergence theorem to obtain

$$\mu^{P_0}(F) = \lim_{k\to\infty} R_u^{\bar{G}_k \cap D}(P_0) = \int_F K(P_0, Q)\, d\nu(Q) = \nu(F),$$

which proves the uniqueness of the measure μ^{P_0} and completes the proof of the theorem.

In the general case considered by Martin the measure in the representation theorem is not unique without a restriction on its support. This is because there may be points $M \in \Delta$ with $\mu^{P}_{\mathscr{K}(\cdot,M)}(\{M\})=0$. (The effect of this is clearly illustrated in the proof of uniqueness we have given in Theorem (4.3).) Martin denotes by Δ_0 the subset of Δ where this occurs and shows that $\mu^{P_0}_{\mathscr{K}(\cdot,M)}(\{M\})=1$ for

$$M \in \Delta_1 = \Delta - \Delta_0.$$

The measure in the representation theorem is unique if it has support in Δ_1. When D is a Lipschitz domain Δ_0 is empty and $\Delta_1 = \partial D$.

Martin characterizes the set $\Delta_1 \subset \Delta$ by showing $M \in \Delta_1$ if and only if $M \in \Delta$ and $\mathscr{K}(P, M)$ is a minimal positive harmonic function—i.e., u is a minimal positive harmonic function if every harmonic v satisfying $0 \leqq v \leqq u$ is a multiple of u. If D is a Lipschitz domain each $K(P, Q)$, $Q \in \partial D$, is a minimal positive harmonic function since if v is harmonic with $0 \leqq v \leqq K(P, Q)$ then $v(P)=v(P_0)K(P, Q)$ by the uniqueness of kernel functions.

5. **Fine, semifine and nontangential limits.** We are interested in studying the relations between the various notions of limit at the boundary for functions defined in Lipschitz domains. The question of the existence of fine limits at the Martin boundary for more general domains is considered in [4] and [8]. Results concerning nontangential limits at the boundary of a Lipschitz domain are considered in [5].

We note that in view of the representation theorem of the previous section the proof given in [5] can be simplified. It is not necessary to reduce the nonnegative case to the bounded case in order to obtain finite nontangential limits a.e. ($d\omega^{P_0}$) for harmonic functions.

Let us recall some definitions. A relatively closed subset B of a Lipschitz domain D is said to be thin at $Q \in \partial D$ if $R^B_{K(\cdot,Q)}(P) < K(P, Q)$ for some $P \in D$. The complement in D of a set which is thin at Q is called a fine neighborhood of Q. A function u on D is said to have fine limit L at $Q \in \partial D$ if for every $\varepsilon > 0$ there is a fine neighborhood N of Q such that $|u(P)-L| < \varepsilon$ for $P \in N$.

A useful characterization of thinness is given by the following lemma.

LEMMA (5.1). *Suppose B is a relatively closed subset of a Lipschitz domain D. Then B is thin at $Q \in \partial D$ if and only if*

$$\lim_{k\to\infty} R^{B\cap\sigma_k}_{K(\cdot,Q)}(P) = 0,$$

where $\sigma_k = \{P : |P-Q| \leqq 2^{-k}\}$.

Proof. Suppose B is thin at Q. The functions $R^{B\cap\sigma_k}_{K(\cdot,Q)}$ decreases to a harmonic function h in D and as in the proof of Lemma (3.4) we obtain $R^B_h = h$. Since h is harmonic with $0 \leqq h \leqq K(\cdot, Q)$ we have $h(P)=h(P_0)K(P, Q)$. Hence, if $h(P_0) \neq 0$, $R^B_h = h$ implies $R^B_{K(\cdot,Q)} = K(\cdot, Q)$. It follows that $h(P_0)=0$ and so $h \equiv 0$.

Conversely, suppose

$$\lim_{k\to\infty} R^{B\cap\sigma_k}_{K(\cdot,Q)}(P) = 0.$$

Then $B \cap \sigma_k$ is thin at Q for some fixed k. If $\tilde{\sigma}_k = \{P : |P-Q| \geqq 2^{-k}\}$, then

$$\begin{aligned}\varphi(P) &= K(P, Q) - R^{B}_{K(\cdot,Q)}(P)\\ &\geqq K(P, Q) - R^{B\cap\sigma_k}_{K(\cdot,Q)}(P) - R^{B\cap\tilde{\sigma}_k}_{K(\cdot,Q)}(P).\end{aligned}$$

Using Lemma (3.3) we then have

$$\limsup_{P\to Q} \varphi(P) \geqq \limsup_{P\to Q} \{K(P, Q) - R^{B\cap\sigma_k}_{K(\cdot,Q)}(P)\} > 0,$$

so B is thin at Q.

The above characterization of thin sets leads to the concept of semithin sets. We recall the definitions. A relatively closed subset B of a Lipschitz domain D is said to be semithin at $Q \in \partial D$ if

$$\lim_{k\to\infty} R^{B\cap\rho_k}(P) = 0,$$

where

$$\rho_k = \{P : 2^{-k-1} \leqq |P-Q| \leqq 2^{-k}\}.$$

Semifine neighborhoods and semifine limits at the boundary then correspond to semithin sets in exactly the same way the analogous concepts correspond to thin sets.

We note that if B is thin at Q then it is semithin at Q. Hence, if u has a fine limit at Q it has the same semifine limit at Q.

We will need certain inequalities which have only been proved for starlike Lipschitz domains. Using such inequalities to obtain results for (semi)thinness for general Lipschitz domains is justified by the following lemma.

LEMMA (5.2). *Suppose D is a Lipschitz domain and $\tilde{D} = \psi(Q, r, s) \cap D$ is a starlike Lipschitz domain. A relatively closed subset B of D is (semi)thin at Q with respect to D if and only if it is (semi)thin at Q with respect to $\tilde{D}$.*

Proof. Let $\tilde{\omega}^P$, $\tilde{R}$ and $\tilde{K}$ in $\tilde{D}$ correspond to ω^P, R and K in D and let $\gamma = D \cap \partial\tilde{D}$. Let B_k denote $(B \cap \rho_k)$ $B \cap \sigma_k$.

We first show that if

$$\lim_{k\to\infty} \tilde{R}^{B_k}_{\tilde{K}(\cdot,Q)}(P) = 0$$

then

$$\lim_{k\to\infty} R^{B_k}_{K(\cdot,Q)}(P) = 0.$$

Note that

$$K(P, Q) = C_0\tilde{K}(P, Q) + \int_\gamma K(Q', Q)\, d\tilde{\omega}^P(Q'), \qquad P \in \tilde{D}.$$

It follows that

$$R^{B_k}_{K(\cdot,Q)}(P) \leqq C_0 \tilde{R}^{B_k}_{\tilde{K}(\cdot,Q)}(P) + \tilde{R}^{B_k}_{\int_\gamma K d\tilde{\omega}}(P) + R^{D-\tilde{D}}_{K(\cdot,Q)}(P), \qquad P \in \tilde{D}.$$

If

$$\limsup_{k\to\infty} R^{B_k}_{K(\cdot,Q)}(P_1) > 0$$

we can choose a subsequence such that

$$\lim_{j\to\infty} R^{B_{k_j}}_{K(\cdot,Q)}(P)$$

is a positive harmonic function $h(P)$ in D, and clearly, $h(P)=h(P_0)K(P, Q)$. From the above we must have

$$h(P_0)K(P, Q_0) \leqq R^{D-\tilde{D}}_{K(\cdot,Q)}(P), \qquad P \in \tilde{D},$$

which is impossible.

Conversely,

$$\tilde{K}(P, Q) \leqq C_0^{-1}K(P, Q), \qquad P \in \tilde{D}.$$

Hence

$$\tilde{R}^{B_k}_{\tilde{K}(\cdot,Q)}(P) \leqq C_0^{-1} R^{B_k}_{K(\cdot,Q)}(P), \qquad P \in \tilde{D},$$

which proves the lemma.

LEMMA (5.3). *If D is a Lipschitz domain and B is a n.t. $\mathscr{B}$ set at $Q \in \partial D$ then B is not semithin at Q.*

Proof. Since we may assume D is a starlike Lipschitz domain, this follows from the last part of the proof of Lemma (3.5).

Certain types of sets are easily seen to be semithin. Let $b(X)$ be a Lip 1 function which defines D near a point $Q_0 \in \partial D$—that is, $D=\{(X, y) : y > b(X)\}$ near Q_0. A nonnegative function $f(X)$ is called a majorant of a set B at Q_0 if

$$B \subset \{(X, y) : b(X) \leqq y \leqq b(X)+f(X)\}.$$

LEMMA (5.4). *Suppose B is a relatively closed subset of a Lipschitz domain D. If B has a majorant f at Q_0 with*

$$\lim_{X\to 0} \frac{f(X)}{|X|} = 0$$

then B is semithin at Q_0.

Proof. By Lemma (2.1),

$$R^{B\cap\rho_k}_{K(\cdot,Q_0)}(P) \leqq C \max\{K(P', Q_0) : P' \in B \cap \rho_k\}\omega^P(\rho_k \cap \partial D), \qquad P \in D.$$

We claim that

$$\max\{K(P', Q_0) : P' \in B \cap \rho_k\} \leqq C\cdot C_k/\omega^{P_0}(\Delta(Q_0, 2^{-k})),$$

where $C_k \to 0$ as $k \to \infty$. To prove this we introduce an auxiliary function.

Let $\tilde{D}=\{(X,y): |X|^2+y^2<1, y>-M|X|\}$ and let $h(P)$ be the harmonic measure in $\tilde{D}$ of the part of $\partial\tilde{D}$ on the sphere. Let $\tilde{D}(Q',r)$ denote the domain obtained from $\tilde{D}$ by shrinking $\tilde{D}$ by a factor r and then translating the origin to Q'. $h(P;Q',r)$, $P\in\tilde{D}(Q',r)$, corresponds to $h(P)$ in $\tilde{D}$. For each $Q'\in\rho_k\cap\partial D$ we have from (2.3), modified as in (2.5), and the maximum principle

$$K(P,Q_0) \leqq CK(A_k,Q_0)h(P;Q',2^{-k-2}),$$

$P\in D\cap\tilde{D}(Q',2^{-k-2})$, where A_k is the point above Q_0 with $|A_k-Q_0|=2^{-k-2}$. From Lemma (3.1) we obtain

$$K(A_k,Q_0) \leqq C/\omega^{P_0}(\Delta(Q_0,2^{-k})).$$

If

$$C_k = \max\{h(0,y): 0<y\leqq 2^{k+2}\cdot\max\{f(X): |X|\leqq 2^{-k}\}\},$$

then $\lim_{k\to\infty} C_k=0$ since $\lim_{X\to 0}(f(X)/|X|)=0$ and $\lim_{y\to 0} h(0,y)=0$. Moreover, $h(P';Q',2^{-k-2})\leqq C_k$ for those $P'\in B\cap\rho_k$ above Q'. Combining results we obtain our assertion and the lemma follows.

THEOREM (5.5). *Suppose D is a Lipschitz domain. If any function in D has a nontangential limit L at $Q\in\partial D$ then it has a semifine limit L at Q. If a nonnegative harmonic function in D has a semifine limit L at $Q\in\partial D$ then it has a nontangential limit L at Q.*

Proof. Suppose u has a nontangential limit L at Q. If $b(X)$ is a Lip 1 function which defines D near $Q=0$, then

$$\Gamma_n = \{(X,y): y>b(X)+(1/n)|X|\}$$

is a nontangential subdomain of D at Q for each n. Hence, given any $\varepsilon>0$ we can truncate each Γ_n near Q so that $|u(P)-L|<\varepsilon$ for $P\in N$, N being the union of the truncated regions. The complement of N is semithin at Q by Lemma (5.4) and it follows that u has semifine limit L at Q.

If the positive harmonic function u does not have nontangential limit L at Q then for some $\varepsilon>0$ there is a sequence of points $\{P_k\}\subset D$ such that P_k approaches Q nontangentially and $|u(P_k)-L|\geqq 2\varepsilon$. By Harnack's principle there is a ball B_k with center P_k and radius proportional to $|P_k-Q|$ such that $|u(P)-L|>\varepsilon$ for $P\in B=\bigcup B_k$. Since the n.t. $\mathscr{B}$ set B is not semithin at Q, u cannot have semifine limit L at Q.

Since the existence at a point of a fine limit implies that of a semifine limit, we have the pointwise result that the existence of fine limits implies that of nontangential limits for positive harmonic functions. Although the converse may not hold pointwise, an almost everywhere implication is true. To prove this we will use the following lemma.

LEMMA (5.6). *Let D be a Lipschitz domain and let F be a closed subset of ∂D. For each $Q\in F$ let $\Gamma(Q)$ denote a nontangential cone in D with vertex at Q and axis* *parallel to the local y-axis, and assume the cones* $\Gamma(Q)$ *have the same size and shape. If* $S=\bigcup_{Q\in F}\Gamma(Q)$, *then the complement of S is thin at a.e.* $(d\omega^{P_0})$ $Q\in F$.

Proof. Let G be the complement of F in ∂D and B be the complement of S in D. We may restrict G and B to points which are under the cones $\Gamma(Q)$, $Q\in F$. For $Q\in G$ let $P(Q)$ denote the point in $D\cap\partial B$ above Q. The intersection of ∂D with a cone oriented downward with vertex at $P(Q)$, axis parallel to the local y-axis and fixed small aperture will contain a disc, denoted $\delta(Q)$, such that the radius of $\delta(Q)$ is a fixed multiple of $|P(Q)-Q|$ and $\delta(Q)\subset G$. For $Q\in G$ we have $\omega^{P(Q)}(\delta(Q))\geqq C>0$ by Lemma (2.1), and for $Q_0\in F$ we have $\min\{K(P(Q),Q_0) : Q\in\delta(Q')\}\geqq CK(P(Q'),Q_0)$ by Harnack's principle.

We will show:

(i) $$R^{B\cap\sigma_k}_{K(\cdot,Q_0)}(P)\leqq C\int_{G\cap\Delta(Q_0,2^{-k})}K(P(Q),Q_0)\,d\omega^P(Q)+R^{B\cap\rho_k}_{K(\cdot,Q_0)}(P),$$
$$Q_0\in F,\ P\in D-(B\cap\sigma_k),$$

(ii) $$R^{B\cap\rho_k}_{K(\cdot,Q_0)}(P_0)\to 0\quad\text{for every } Q_0\in F \text{ such that } \int_G K(P(Q),Q_0)d\omega^{P_0}(Q)<\infty,$$

and

(iii) $$\int_G K(P(Q),Q_0)\,d\omega^{P_0}(Q)<\infty\quad\text{for a.e. } (d\omega^{P_0})Q_0\in F.$$

The theorem is clearly a consequence of (i), (ii) and (iii).

Let us consider (i). For $Q_0\in F$ and $Q'\in\Delta(Q_0,2^{-k-1})$ we have

$$\int_{G\cap\Delta(Q_0,2^{-k})}K(P(Q),Q_0)\,d\omega^{P(Q')}(Q)$$
$$\geqq\int_{\delta(Q')}K(P(Q),Q_0)\,d\omega^{P(Q')}(Q)$$
$$\geqq CK(P(Q'),Q_0)\omega^{P(Q')}(\delta(Q'))\geqq CK(P(Q'),Q_0).$$

This shows that $K(P,Q_0)$ is majorized on $D\cap\partial(B\cap\sigma_{k+1})$ by

$$C\int_{G\cap\Delta(Q_0,2^{-k})}K(P(Q),Q_0)\,d\omega^P(Q).$$

At points $P\in\partial(B\cap\rho_k)$, $R^{B\cap\rho_k}_{K(\cdot,Q_0)}(P)=K(P,Q_0)$ and (i) follows.

We will show that

$$\int_G K(P(Q),Q_0)\,d\omega^{P_0}(Q)<\infty$$

implies $|P(Q)-Q|/|Q-Q_0|\to 0$ as $Q\to Q_0$, $Q\in G$, $Q_0\in F$. Suppose, on the contrary, that there exist $Q_k\to Q_0$, $Q_k\in G$, with $|P(Q_k)-Q_k|/|Q_k-Q_0|\geqq\varepsilon>0$.

Assuming as we may that the discs $\delta(Q_k)$ are disjoint we have

$$\int_G K(P(Q), Q_0)\, d\omega^{P_0}(Q) \geqq \sum_{k=1}^{\infty} \int_{\delta(Q_k)} K(P(Q), Q_0)\, d\omega^{P_0}(Q)$$
$$\geqq C \sum_{k=1}^{\infty} K(P(Q_k), Q_0)\omega^{P_0}(\delta(Q_k)).$$

By Lemma (3.1) $\omega^{P_0}(\delta(Q_k)) \geqq C\omega^{A_k}(\delta(Q_k))\omega^{P_0}(\Delta_k)$, where Δ_k is the smallest disc with center Q_0 containing $\delta(Q_k)$ and A_k is the point above Q_0 with $|A_k - Q_0|$ equal to the radius of Δ_k. Since $|P(Q_k) - Q_k| \geqq \varepsilon |Q_0 - Q_k|$, $\omega^{A_k}(\delta(Q_k)) \geqq C\omega^{P(Q_k)}(\delta(Q_k))$ and $K(P(Q_k), Q_0) \geqq CK(A_k, Q_0)$ by Harnack's principle. Hence

$$\int_G K(P(Q), Q_0)\, d\omega^{P_0}(Q) \geqq C \sum_{k=1}^{\infty} K(A_k, Q_0)\omega^{P_0}(\Delta_k).$$

However, by (2.4) each term of this sum exceeds a fixed positive number and this contradicts the assumption that $\int_G K(P(Q), Q_0)\, d\omega^{P_0}(Q) < \infty$. Lemma (5.4) now completes the proof of (ii).

To prove (iii), we note that

$$\int_F \left[\int_G K(P(Q), Q_0)\, d\omega^{P_0}(Q)\right] d\omega^{P_0}(Q_0) = \int_G \left[\int_F K(P(Q), Q_0)\, d\omega^{P_0}(Q_0)\right] d\omega^{P_0}(Q)$$
$$= \int_G [\omega^{P(Q)}(F)]\, d\omega^{P_0}(Q)$$
$$\leqq \omega^{P_0}(G).$$

This completes the proof of Lemma (5.6).

THEOREM (5.7). *Suppose u is defined in a Lipschitz domain D and has a finite nontangential limit $u(Q)$ at each point of a set $E \subset \partial D$. Then u has fine limit $u(Q)$ at a.e. $(d\omega^{P_0})$ $Q \in E$.*

Proof. Given $\varepsilon > 0$ let $\Gamma(Q)$, $Q \in E$, be a nontangential cone in D with vertex at Q, fixed aperture, axis parallel to the local y-axis and truncated so that $|u(P) - u(Q)| < \varepsilon$ for $P \in \Gamma(Q)$. By truncating rationally and dividing E into a countable number of subsets we may assume that all cones $\Gamma(Q)$ have the same height. For any closed subset F of E the complement of $\bigcup_{Q \in F} \Gamma(Q)$ is thin at a.e. $(d\omega^{P_0})$ $Q \in F$ by the previous lemma. Hence, the complement of $\bigcup_{Q \in E} \Gamma(Q)$ is thin at a.e. $Q \in E$. For $Q \in E$ let $N(Q)$ be the union of all cones $\Gamma(Q')$ with $Q' \in E$ and $\Gamma(Q') \cap \Gamma(Q) \neq \varnothing$. Then $|u(P) - u(Q)| < 3\varepsilon$ for $P \in N(Q)$ and since $D - N(Q)$ is thin at a.e. $(d\omega^{P_0})$ $Q \in E$, the theorem follows.

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